

Lecture 1B: Entanglement Measures as Correlators

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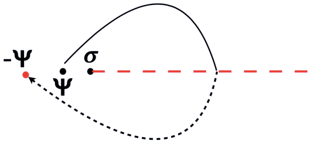
1. Twist Fields in Quantum Field Theory

- Whenever there is an internal symmetry in QFT there generally is a symmetry field associated to it.
- A well-known example is the Ising model.

$$\text{Minimal Model of CFT}: \quad \{\mathbf{1}, \varepsilon, \sigma\}$$

- The massive perturbation of this theory, the Ising field theory, also contains a Majorana fermion Ψ.
- With respect to Ψ the theory divides into two sectors (i.e. $\mathbb{Z}_2$ symmetry): fields that are local w.r.t. the fermion $\{\mathbf{1}, \varepsilon, \mu\}$ and fields that are semi-local $\{\sigma\}$ [Yurov & Zamolodchikov'90].



$$\varphi(\mathbf{y})\Psi(\mathbf{x}) = \begin{cases} \Psi(\mathbf{x})\varphi(\mathbf{y}) & \textbf{for} \quad \mathbf{x^1} > \mathbf{y^1} \\ \omega\Psi(\mathbf{x})\varphi(\mathbf{y}) & \textbf{for} \quad \mathbf{x^1} < \mathbf{y^1} \end{cases}$$

- Twist fields are non-local w.r.t. other fields in the theory but

Poll 1

Is there a similar or related picture in the spin chain setting?
Think of the Ising chain for instance....

A. No, twist fields only really make sense in QFT.

B. Yes, twist fields are related to local operators in the chain.

Poll 2

When were branch point twist fields first “discovered”?

A. They were first introduced by Calabrese and Cardy in their famous 2004 paper.

B. They have been know for ages!

C. They were first introduced by Cardy, OC-A and Doyon in their (less famous) 2008 paper.

2. Branch Point Twist Fields in a Nutshell

- It has been known for a while that a special field $\mathcal{T}$ may be associated to the $\mathbb{Z}_n$ symmetry of an orbifolded CFT constructed as n cyclicly connected copies of a given CFT. [Knizhnik'87; Kac & Wakimoto'90; Bouwknegt'96; Borisov et al.'98]

Twist Field Conformal Dimension(s)

$$\Delta_{\mathcal{T}} = \frac{c}{24}\left(n - \frac{1}{n}\right) \qquad \Delta_{:\mathcal{T}\phi:} = \frac{c_{\text{eff}}}{24}\left(n - \frac{1}{n}\right)$$

- In the context of entanglement a field Φ of dimension $\frac{\Delta_{\mathcal{T}}}{n}$ was first identified in [Calabrese & Cardy'04]. In this work, this field was interpreted as associated to a conical singularity in $\mathbb{C}$ (the branch point(s) of our Riemann surface).
- In [Cardy, OC-A & Doyon'08] we introduced a field $\mathcal{T}$ and its conjugate $\tilde{\mathcal{T}}$. We called this field branch point twist field and described it as the symmetry field associated to cyclic permutations in a replica theory. We showed that it had dimension $\Delta_{\mathcal{T}}$.
- The composite twist field appeared in the study of the EE of non-unitary CFTs [Bianchini et al.'15; Bianchini & Ravanini'16]

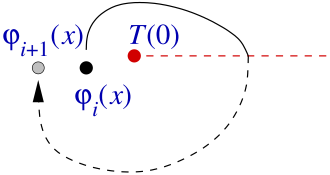
3. BPTFs from Exchange Relations in QFT

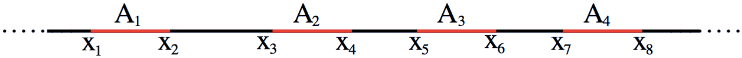
- Let φ be a local field of a given QFT. Consider a replica theory where n copies φ_i, $i = 1, \ldots, n$ exist and $i + n \equiv i$.
- Two Branch Point Twist Fields may be defined which are characterized by the following commutation relations:

$$\begin{aligned}
\varphi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\varphi_{i+1}(y) \qquad x^1 > y^1, \\
\varphi_i(y)\mathcal{T}(x) &= \mathcal{T}(x)\varphi_i(y) \qquad x^1 < y^1,
\end{aligned}$$

$$\begin{aligned}
\varphi_i(y)\mathcal{T}^\dagger(x) &= \mathcal{T}^\dagger(x)\varphi_{i-1}(y) \qquad x^1 > y^1, \\
\varphi_i(y)\mathcal{T}^\dagger(x) &= \mathcal{T}^\dagger(x)\varphi_i(y) \qquad x^1 < y^1.
\end{aligned}$$

- $\mathcal{T}$ implements the cyclic permutation symmetry $i \mapsto i + 1$ and $\mathcal{T}^\dagger$ implements the inverse map $i \mapsto i - 1$.

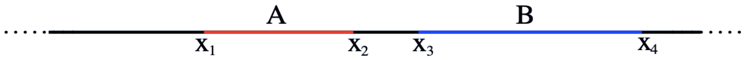




General Statement 1: The Rényi entropy of k disconnected regions in 1D QFT, which includes CFT and massive QFT, in a (replica) pure state $|\Psi\rangle_n$ can be obtained from a replica $2k$-point function involving as many twist and anti-twist field insertions as boundary points.

Rényi Entropy of Multiple Disconnected Regions

$$S_n(\{x_i\}) = \frac{1}{1-n} \log \left(\varepsilon^{4k\Delta_{\mathcal{T}_n}} \langle \Psi | \mathcal{T}(x_1) \mathcal{T}^\dagger(x_2) \cdots \mathcal{T}(x_{2k-1}) \mathcal{T}^\dagger(x_{2k}) | \Psi \rangle_n \right)$$



General Statement 2: The Replica Logarithmic Negativity of two connected non-complementary regions A and B in 1D QFT, in a (replica) pure state $|\Psi\rangle_n$ can be obtained from a replica 4-point function involving as many twist and anti-twist field insertions as boundary points.

Replica Logarithmic Negativity of Connected Regions

$$\mathcal{E}_n(x_1,x_2,x_3,x_4)=\frac{1}{1-n}\log\left(\varepsilon^{4k\Delta_{\mathcal{T}_n}}\langle\Psi|\mathcal{T}(x_1)\mathcal{T}^\dagger(x_2)\mathcal{T}^\dagger(x_3)\mathcal{T}(x_4)|\Psi\rangle_n\right)$$

Poll 3

How is this useful? Do we have to compute this correlation functions on the n-sheeted manifold?

A. It is useful but we still have to use special methods to map to $\mathbb{R}_2$

B. These are now correlation functions on $\mathbb{R}_2$ so they can be computed by "standard" methods. That's really the point of twist fields!

6. Revisiting EE Results from BPTFs

Logarithmic Scaling in CFT ($\xi \gg \ell$)

$$\mathrm{Tr}_{\mathcal{A}}(\rho_A^n) = \varepsilon^{4\Delta_{\mathcal{T}_n}} \langle 0|\mathcal{T}(0)\mathcal{T}^\dagger(\ell)|0\rangle_n \quad \Rightarrow \quad S_n(\ell) = \frac{c(n+1)}{6n}\log\frac{\ell}{\varepsilon}$$

- Logarithmic growth follows from the power-law scaling of two-point functions in CFT.

Saturation for $\xi \ll \ell$

$$\lim_{\ell\to\infty} \mathrm{Tr}_{\mathcal{A}}(\rho_A^n) = \varepsilon^{4\Delta_{\mathcal{T}_n}} \langle 0|\mathcal{T}|0\rangle_n^2 \;\Rightarrow\; S_n(\ell) \sim \frac{c(n+1)}{6n}\log\frac{\varepsilon}{m} + 2U_n$$

- Saturation follows from factorization of correlators at large distances and from the scaling properties of vacuum expectation values in 2D QFT: ${}_n\langle 0|\mathcal{T}|0\rangle_n = m^{2\Delta_{\mathcal{T}}} a_n$ and $U_n = \frac{\log a_n}{1-n}$.
- The exponentially decaying corrections that we saw earlier can also be obtained from this approach, but require analysing subleading terms in the expansion of the two-point function. We will come back to this later.

Poll 4

What do the exponentially decaying corrections to saturation mean (remember they looked like $-1/8K_0(2m_a\ell)$)?

A. The crossover between CFT and gapped QFT behaviour.

B. In gapped systems there is a maximum amount of entanglement that can be accrued for a large region. If the region is not very large but still larger than ξ , entanglement is reduced.

C. The entanglement entropy contains information about the mass spectrum of 1+1D QFT.

Poll 5

What is the leading field in the OPE of $\mathcal{T}(x_1)\mathcal{T}(x_2)$?

A. The identity.

B. The same branch point twist field.

C. A possibly different branch point twist field.

7. Revisiting LN Results from BPTFs

Logarithmic Negativity of Adjacent Regions $\xi \gg \ell$

$$\mathrm{Tr}_{\mathcal{A}\cup\mathcal{B}}(\rho_{A\cup B}^{T_B})^n = \varepsilon^{4\Delta_{\mathcal{T}}+2\Delta_{\mathcal{T}^2}}{}_n\langle 0|\mathcal{T}(-\ell_1)(\mathcal{T}^\dagger)^2(0)\mathcal{T}(\ell_2)|0\rangle_n \quad \Rightarrow$$

$$\mathcal{E}_n(\ell_1,\ell_2) = -2\Delta_{\mathcal{T}^2}\log\frac{\ell_1\ell_2}{\varepsilon(\ell_1+\ell_2)} - 4\Delta_{\mathcal{T}}\log\frac{\ell_1+\ell_2}{\varepsilon} + \log(\mathcal{C}_{\mathcal{T}\mathcal{T}^2\mathcal{T}})$$

- Logarithmic growth follows from the power-law scaling of three-point functions in CFT. $\mathcal{T}^2 = \mathcal{T}_{\frac{n_e}{2}} \otimes \mathcal{T}_{\frac{n_e}{2}}$ [Calabrese, Cardy & Tonni'12] where $\Delta_{\mathcal{T}^2}$ is twice $\Delta_{\mathcal{T}}$ up to $n \mapsto \frac{n}{2}$.

Logarithmic Negativity of Adjacent Regions $\xi \ll \ell$

$$\lim_{\ell_1,\ell_2\to\infty} \varepsilon^{4\Delta_{\mathcal{T}}+2\Delta_{\mathcal{T}^2}}{}_n\langle 0|\mathcal{T}(-\ell_1)(\mathcal{T}^\dagger)^2(0)\mathcal{T}(\ell_2)|0\rangle_n \quad \Rightarrow$$

$$\mathcal{E}_n = 2(\Delta_{\mathcal{T}^2} + 2\Delta_{\mathcal{T}})\log m\varepsilon + 2E_n$$

- Saturation follows again from clustering and the scaling: ${}_n\langle 0|\mathcal{T}|0\rangle_n = m^{2\Delta_{\mathcal{T}}} a_n$ and ${}_n\langle 0|\mathcal{T}^2|0\rangle_n = m^{2\Delta_{\mathcal{T}^2}} b_n^2$ so $E_n = \log b_n a_n$. There are exponentially decaying corrections. These require more work!

8. LN Results from BPTFs: Four-Point Function

- We end this section by mentioning that the full four-point function involved in the standard definition of the LN and in the EE of two disconnected regions have also been studied in CFT.

Disconnected Regions in CFT

$$\mathrm{Tr}_{\mathcal{A}\cup\mathcal{B}}(\rho_{A\cup B}^{T_B})^n = \varepsilon^{8\Delta_{\mathcal{T}_n}}\langle 0|\mathcal{T}(\ell_1)\mathcal{T}^\dagger(\ell_2)\mathcal{T}^\dagger(\ell_3)\mathcal{T}(\ell_4)|0\rangle_n \quad \text{LN}$$

$$\mathrm{Tr}_{\mathcal{A}\cup\mathcal{B}}(\rho_{A\cup B})^n = \varepsilon^{8\Delta_{\mathcal{T}_n}}\langle 0|\mathcal{T}(\ell_1)\mathcal{T}^\dagger(\ell_2)\mathcal{T}(\ell_3)\mathcal{T}^\dagger(\ell_4)|0\rangle_n \quad \text{EE}$$

- The most extensive study is due to [Calabrese, Cardy & Tonni'10-13] and looked at the compactified free boson.
- Even for this theory, the analytic continuation in n is non-trivial and is not fully understood analytically.