Lecture 2B: Form Factor Solution Procedure

Olalla A. Castro-Alvaredo

School of Mathematics, Computer Science and Engineering
Department of Mathematics
City, University of London

Galileo Galilei Institute, Arcetri, Florence
8-19 February 2021

1. Solving Watson's Equations

- For diagonal theories, the solution procedure is standard and easy to generalize to BPTFs.
- Since the replicas are mutually independent $S_{\mu_1\mu_2}(\theta) = (S_{a_1 a_2}(\theta))^{\delta_{ij}}$ where $a_{1,2}$ are particle indices and i, j are copy indices.
- The usual starting point is finding a minimal solution to the equations for two-particle form factors $F_{\mathbf{min}}^{\mathcal{T}|\mu_1\mu_2}(\theta_1, \theta_2) =: F_{\mathbf{min}}^{\mathcal{T}|\mu_1\mu_2}(\theta)$.
- Watson's equations can be written as:

$$F_{\mathbf{min}}^{\mathcal{T}|\mu_1\mu_2}(\theta) = S_{\mu_1\mu_2}(\theta) F_{\mathbf{min}}^{\mathcal{T}|\mu_2\mu_1}(-\theta) = F_{\mathbf{min}}^{\mathcal{T}|\mu_2\mu_1}(-\theta + 2\pi i n)$$

- For IQFT there is a systematic method for solving this type of equation. The starting point is an integral representation of the two-particle scattering matrix:

$$S_{ab}(\theta) = \exp\left[\int_0^\infty \frac{dt}{t} f_{ab}(t) \sin\frac{\theta t}{i\pi}\right] \quad \Rightarrow$$

$$F_{\mathbf{min}}^{\mathcal{T}|ab}(\theta) = \mathcal{N} \exp\left[\int_0^\infty \frac{dt}{t} \frac{f_{ab}(t)}{\sinh(n\theta)} \sin^2 \frac{t(i\pi n - \theta)}{2\pi}\right]$$

for a, b in the same copy.

2. Towards a Full Solution

- Once we have a minimal two-particle solution it is easy to show that a function of the form

$$F_{\mathbf{min}}^{\mathcal{T}|a_1\ldots a_k}(\theta_1,\ldots,\theta_k) \propto \prod_{1\leq i<j\leq k} F_{\mathbf{min}}^{\mathcal{T}|a_i a_j}(\theta_i-\theta_j)$$

  solves Watson's equations for any particle number. So a function of this type must be part of the general solution.

- We must now incorporate the necessary poles. Here we will just look at the case without bound states.

- For two particles the following function is what we need:

Two-Particle Form Factor

$$F_2^{\mathcal{T}|a\bar{a}}(\theta) = \frac{\langle\mathcal{T}\rangle \sin\frac{\pi}{n}}{2n\sinh\left(\frac{i\pi-\theta}{2n}\right)\sinh\left(\frac{i\pi+\theta}{2n}\right)} \frac{F_{\mathbf{min}}^{\mathcal{T}|a\bar{a}}(\theta)}{F_{\mathbf{min}}^{\mathcal{T}|a\bar{a}}(i\pi)} \quad \text{with} \quad F_0^{\mathcal{T}} = \langle\mathcal{T}\rangle$$

- Satisfies all equations (exercise!). Has simple poles at $\theta = i\pi$ and $\theta = i\pi(2n-1)$ on the extended physical sheet $\mathrm{Im}(\theta)\in[0,2\pi n]$.The function containing the poles is even and $2\pi n$-periodic in θ.

3. Form Factor Ansatz

- We need a function with the following features:
  1) Includes the product of minimal form factors.
  2) Has the correct number of kinematic poles.
  3) Apart from the minimal form factors, any other function involved must be symmetric $2\pi n$-periodic in all rapidities.

General Ansatz for Particles in the Same Copy

$$F_k^{\mathcal{T}|a_1\ldots a_k}(\theta_1,\ldots,\theta_k) = H_k^{a_1\ldots a_k} Q_k^{a_1\ldots a_k}(x_1,\ldots,x_k)$$

$$\times \prod_{1\leq i<j\leq k} \frac{F_{\mathbf{min}}^{\mathcal{T}|a_i a_j}(\theta_i-\theta_j)}{\left[(x_i - e^{\frac{i\pi}{n}} x_j)(x_j - e^{\frac{i\pi}{n}} x_i)\right]^{\delta_{a_i,\overline{a_j}}}}$$

- The functions Q_k must be combinations of elementary symmetric polynomials on the variables $x_i = e^{\frac{\theta_i}{n}}$. H_k are θ_i-independent.
- Additional poles are present if there are bound states. They will take the form: $(x_i - e^{\frac{i\pi u_{ab}^c}{n}} x_j)(x_j - e^{\frac{i\pi u_{ab}^c}{n}} x_i)$.

4. Tackling Different Copies

- All form factors can be related to a form factor of particles in the same copy. Also the form factors of $\mathcal{T}^\dagger$ are related to those of $\mathcal{T}$.
- This follows from permutation symmetry among copies and the properties of the BPTFs. For example:

Useful Properties

$$
\begin{aligned}
F_2^{\mathcal{T}|(a,i)(b,i+k)}(\theta) &= F_2^{\mathcal{T}|(a,j)(b,j+k)}(\theta) = F_2^{\mathcal{T}|(a,1)(b,1+k)}(\theta) \\
F_2^{\mathcal{T}|(a,1)(b,j)}(\theta) &= F_2^{\mathcal{T}|(b,1)(a,1)}(2\pi(j-1)i-\theta) \quad \text{for} \quad j>1 \\
F_2^{\mathcal{T}|(a,i)(b,j)}(\theta) &= F_2^{\mathcal{T}^\dagger|(a,n-i)(b,n-j)}(\theta) \\
F_2^{\mathcal{T}^\dagger|(a,1)(b,j)}(\theta) &= F_2^{\mathcal{T}|(a,1)(b,1)}(2\pi(j-1)i+\theta) \\
F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k) &= F_k^{\mathcal{T}|a_1\ldots a_k}(\theta_1^{j_1},\theta_2^{j_2},\ldots,\theta_k^{j_k}), \quad j_1>j_2\cdots>j_k
\end{aligned}
$$

- Here $\mu_p=(a_p,j_p)$ and $\theta^j=\theta+2\pi i(j-1)$.
- A special feature of the twist field form factors is that they must all vanish at $n=1$ (except for $\langle\mathcal{T}\rangle\mapsto 1$).

5. Application: Sum Manipulation

- In Lecture 3A we will study the two-point function ${}_n\langle 0|\mathcal{T}(0)\mathcal{T}^\dagger(\ell)|0\rangle_n$ and its form factor expansion. The sums below will feature:
- One-particle form factor:

$$\sum_{a=1}^{N}\sum_{j=1}^{n} F^{\mathcal{T}|\mu}[F^{\mathcal{T}^\dagger|\mu}]^* = n\sum_{a=1}^{N}\left|F^{\mathcal{T}|(a,1)}\right|^2 = n\sum_{a=1}^{N}\left|F^{\mathcal{T}|a}\right|^2$$

- Two-particle form factor

$$\sum_{a_1,a_2=1}^{N}\sum_{j_1,j_2=1}^{n} F^{\mathcal{T}|\mu_1\mu_2}(\theta)\left[F^{\mathcal{T}^\dagger|\mu_1\mu_2}(\theta)\right]^*$$

$$= n\sum_{a_1,a_2=1}^{N}\sum_{j=1}^{n} F^{\mathcal{T}|(a_1,1)(a_2,j)}(\theta)\left[F^{\mathcal{T}^\dagger|(a_1,1)(a_2,j)}(\theta)\right]^*$$

$$= n\sum_{a_1,a_2=1}^{N}\left[\left|F^{\mathcal{T}|(a_1,1)(a_2,1)}(\theta)\right|^2 + \sum_{j=2}^{n}\left|F^{\mathcal{T}|(a_2,1)(a_1,1)}(2\pi i(j-1)-\theta)\right|^2\right]$$

$$= n\sum_{a_1,a_2=1}^{N}\left[\left|F^{\mathcal{T}|a_1a_2}(\theta)\right|^2 + n\sum_{j=1}^{n-1}\left|F^{\mathcal{T}|a_2a_1}(2\pi ij-\theta)\right|^2\right]$$