

Lecture 3A: Exponential Corrections to Saturation

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1. Exponential Corrections to Saturation

- In this last two hours I will introduce two applications of the techniques I have presented.
- In this lecture we will consider corrections to the saturation of the entanglement entropy of large subsystems in gapped systems.
- Saturation is a feature of the entanglement entropy of 1+1D gapped systems that has been mathematically proven by [Hastings'07] and also shown numerically [Vidal, Latorre, Rico & Kitaev'03] and analytically [Calabrese & Cardy'04]
- In the context of BPTFs saturation follows simply from clustering of correlators:

$$\lim_{\ell\to\infty} {}_n\langle 0|\mathcal{T}(0)\mathcal{T}^\dagger(\ell)|0\rangle_n = \langle\mathcal{T}\rangle^2 \quad \Rightarrow \quad \lim_{\ell\to\infty} S_n(\ell) = -\frac{c(n+1)}{6n}\log(m\varepsilon)+U_n$$

with $\langle\mathcal{T}\rangle = m^{2\Delta_\mathcal{T}} a_n$ and $U_n = 2(1-n)^{-1}\log a_n$.

- In the form factor context, this is just the first (leading) term in the form factor expansion. What will other terms tell us?

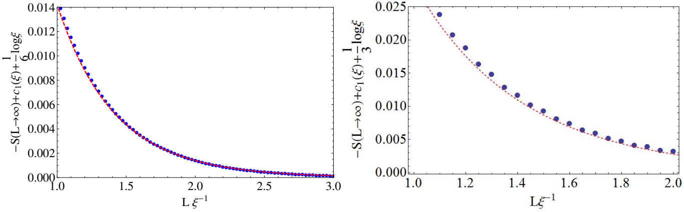
2. Exponential Corrections to Saturation

- In [Cardy, OC-A & Doyon'08] we computed the leading correction to saturation of the entanglement entropy.

Universal Correction to Saturation

$$S(\ell) = -\frac{c}{3}\log(m_1\varepsilon) + 2U_1 - \frac{1}{8}\sum_{\alpha=1}^{N} K_0(2\ell m_\alpha) + O\left(e^{-3m_1\ell}\right)$$

m_α is the mass spectrum, $m_1 \propto \xi^{-1}$ is the smallest mass, N is the number of particles in the spectrum.



Ising/sine-Gordon

3. Two-Point Function Expansion

- Recall that

$$S(\ell) = -\lim_{n\to 1} \frac{\partial h(n)}{\partial n} \quad \text{with} \quad h(n) = \varepsilon^{4\Delta_\mathcal{T}}\, {}_n\langle 0|\mathcal{T}(0)\mathcal{T}^\dagger(\ell)|0\rangle_n$$

- The first few terms in our expansion are

$${}_n\langle 0|\mathcal{T}(0)\mathcal{T}^\dagger(\ell)|0\rangle_n = \langle\mathcal{T}\rangle_n^2 + \sum_\mu \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, F_1^{\mathcal{T}|\mu}(F_1^{\mathcal{T}^\dagger|\mu})^* e^{-\ell e_\mu(\theta)}$$

$$+\frac{1}{2}\sum_{\mu_1\mu_2}\int_{-\infty}^{\infty}\frac{d\theta_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\theta_2}{2\pi}\, F_2^{\mathcal{T}|\mu_1\mu_2}(\theta_1-\theta_2)(F_2^{\mathcal{T}^\dagger|\mu_1\mu_2}(\theta_1-\theta_2))^* e^{-\ell(e_{\mu_1}(\theta_1)+e_{\mu_2}(\theta_2))}$$

$$+\cdots$$

with $e_\mu(\theta) = m_\mu \cosh\theta$.

- From here onwards we will consider a theory with a single particle in the spectrum.

4. Beyond Saturation: One-Particle Form Factor

- For a theory with a single particle the one-particle form factor contribution can be written simply as

$$n\,|F_1(n)|^2 \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-\ell m \cosh\theta} = \frac{n}{\pi}\,|F_1(n)|^2\, K_0(m\ell).$$

with $F_1(n) := F_1^{\mathcal{T}|1}$

- When the one-particle form factor is non-zero (there are many theories where it is zero by symmetry!), it provides the leading correction to saturation of the two-point function (hence to the Rényi entropies) .
- However it vanishes under differentiation w.r.t. n and limit $n \to 1$.
- This is because $F_1(n) \propto \mathcal{O}((n-1))$ for $n \to 1$ (we know that $F_1(1) = 0$).
- This means that the one-particle form factors (if they are non-vanishing) will provide the leading correction to the Rényi entropies but no contribution to the EE.

5. Two-Particle Form Factors

- Recall our two-particle form factor sum

$$n\sum_{j=1}^{n}(F_2^{1j}(\theta,n))^*(\tilde{F}_2^{1j}(\theta,n)) = n\left|F_2^{11}(\theta,n)\right|^2 + n\sum_{j=2}^{n}\left|F_2^{11}(-\theta+2\pi i(j-1),n)\right|^2$$

$$= n(\left|F_2^{11}(\theta,n)\right|^2 - \left|F_2^{11}(-\theta,n)\right|^2) + n\sum_{j=0}^{n-1}\left|F_2^{11}(-\theta+2\pi ij,n)\right|^2$$

where we simplified $F_2^{\mathcal{T}|11}(\theta) := F_2^{11}(\theta,n)$.

- The derivative at $n=1$ of the first term will be zero because $F_2^{11}(\theta,1) = F_2^{11}(\theta,1)^* = 0$. So it will contribute to the Rényi entropies but not to the entanglement entropy.

6. Leading Correction to the Entanglement Entropy

- In summary, we need to compute

$$-\frac{1}{4}\lim_{n\to 1}\frac{\partial}{\partial n}\left(\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\int_{-\infty}^{\infty}\frac{d\beta}{2\pi}n\sum_{j=0}^{n-1}\left|F_2^{11}(-\theta+2\pi ij,n)\right|^2 e^{-2m\ell\cosh\frac{\theta}{2}\cosh\frac{\beta}{2}}\right)$$

with $\theta=\theta_1-\theta_2$ and $\beta=\theta_1+\theta_2$.

- The integral in β can be carried out giving a Bessel function. So, we end up with:

$$-\lim_{n\to 1}\frac{\partial}{\partial n}\left(n\int_{-\infty}^{\infty}\frac{d\theta}{(2\pi)^2}\sum_{j=0}^{n-1}\left|F_2^{11}(-\theta+2\pi ij,n)\right|^2 K_0(2m\ell\cosh\frac{\theta}{2})\right)$$

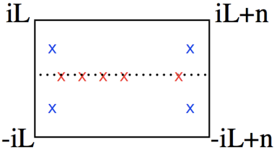
- In order to take the derivative, we need to somehow get rid of the sum up to $n-1$.
- A well-known way of doing this is to use the cotangent trick.

7. Cotangent Trick

- The idea is that the sum may be replaced by a contour integral

$$\frac{1}{2\pi i}\oint dz \pi \cot(\pi z) s(z,\theta,n)$$

with $s(z,\theta,n) = \left|F_2^{11}(-\theta + 2\pi i z, n)\right|^2$, in such a way that the sum of the residues of poles of the cotangent enclosed by the contour reproduces the original sum.



- The red crosses are the poles of the cotangent at $z = 1, 2, \ldots, n-1$. The blue crosses represent other poles due to the kinematic poles of the function $s(z,\theta,n)$ at $z = \frac{1}{2} \pm \frac{\theta}{2\pi i}$ and $z = n - \frac{1}{2} \pm \frac{\theta}{2\pi i}$.
- We shift $iL \to iL - \epsilon$ so as to avoid the pole at $z = n$ and include $z = 0$ (but this does not affect the result).

8. Contributions to the Integral

- Since $s(z,\theta,n)$ decays exponentially as $\mathrm{Im}(z) \to \pm\infty$ so we can show that the contributions to the contour integral of the horizontal segments vanish.
- The contribution of the vertical segments can be written as:

$$-\frac{1}{4\pi i}\int_{-\infty}^{\infty}(S(\theta-\beta)S(\theta+\beta)-1)\coth\frac{\beta}{2}s(\beta,\theta,n)d\beta$$

  where $\beta = 2\pi i z$ and $S(\theta)$ is the S-matrix. Here we used the property $s(z+n,\theta,n) = S(\theta-2\pi i z)S(\theta+2\pi i z)s(z,\theta,n)$.
- Note that this is zero for free theories. Its derivative at $n=1$ is zero for similar reasons as before.
- Finally we are left with the contributions from the residues of the kinematic poles. They give:

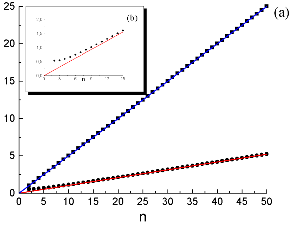
$$\tanh\frac{\theta}{2}\mathrm{Im}\left(F_2^{11}(-2\theta+i\pi,n)-F_2^{11}(-2\theta+2\pi i n-i\pi,n)\right)$$

9. Almost there...

- The only two-particle contribution to the derivative comes from:

$$\mathrm{Im}\left(F_2^{11}(-2\theta+i\pi,n) - F_2^{11}(-2\theta+2\pi i n - i\pi,n)\right)\tanh\frac{\theta}{2}$$

- Based on previous observations, it would seem that this should be zero as $F_2^{11}(\theta,1)=0$. However, something special happens to this function as $n \to 1$ and $\theta \to 0$ simultaneously.



- The sum $n\sum_{j=0}^{n-1}\left|F_2^{11}(-\theta+2\pi i j,n)\right|^2$ for $\theta=0$ in the Ising model (blue) and the sinh-Gordon model (red).

10. Delta Function

- Near $\theta = 0$ and $n = 1$:

$$\text{Im}\left(F_2^{11}(-2\theta + i\pi, n) - F_2^{11}(-2\theta + 2\pi i n - i\pi, n)\right) \tanh\frac{\theta}{2}$$
$$\sim -\frac{1}{2}\left(\frac{i\pi(n-1)}{2(\theta + i\pi(n-1))} - \frac{i\pi(n-1)}{2(\theta - i\pi(n-1))}\right) \sim \frac{\pi^2(n-1)}{2}\delta(\theta).$$

- Putting this result back into the θ integral and differentiating w.r.t. n we obtain the two-particle form factor contribution:

$$-\frac{1}{8}K_0(2m\ell)$$

- The result is striking for its simplicity. From the derivation we see that it follows from the kinematic pole structure of the form factors, which is universal.
- For this reason the same result can be found even for non-integrable 1+1 dimensional models [Doyon'09].
- This kind of phenomenon extends to higher terms in the form factor expansion. We did a full analysis for the Ising model in [OC-A & Doyon'09] (with and without a boundary).