

Lecture 3B: Entanglement Dynamics and Oscillations

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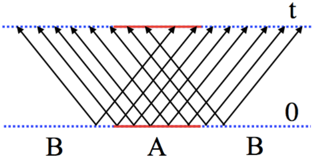
1. Entanglement Dynamics

- What is it? The time evolution of some measure of entanglement after a quantum quench.
- A system prepared initially in the ground state of a hamiltonian $H(\lambda_0)$ time-evolves unitarily with a different hamiltonian $H(\lambda)$.
- For one-dimensional quantum systems and for a particular measure, the entanglement entropy $S(t)$, a lot is understood



$\Rightarrow$ Entanglement Dynamics [Calabrese & Cardy'05'06]

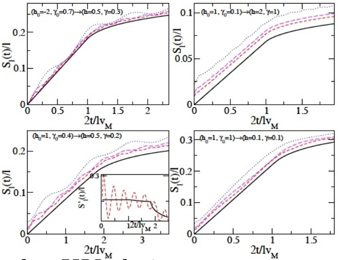
- Argument: Quasi-particle pairs propagate with opposite momenta after the quench

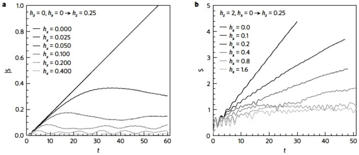
$$S(t) = \mathcal{A}\, t|_{t\leq \frac{\ell}{2v}} + \mathcal{B}\, \ell|_{t>\frac{\ell}{2v}}$$

where $\mathcal{A}$ and $\mathcal{B}$ are theory-dependent, ℓ is the length of A, and v the propagation velocity [Alba & Calabrese'17]

2. Evidence and Further Studies

The quasi-particle picture has wide support. [Fagotti & Calabrese'08]





For the XY chain

$$H = -\sum_{j=1}^{N}\left[a\,\sigma_j^x\sigma_{j+1}^x + b\,\sigma_j^y\sigma_{j+1}^y + \frac{h}{2}\sigma_j^z\right]$$

with $a+b=1/2$.
For $b=0$ Ising chain.
But other things can also happen...

For the Ising chain with both longitudinal and transversal magnetic field showing confinement [Kormos, Collura, Takács & Calabrese'16]

$$H = -J\sum_{j=1}^{N}\left[\sigma_j^x\sigma_{j+1}^x + h_x\sigma_j^x + h_z\sigma_j^z\right]$$

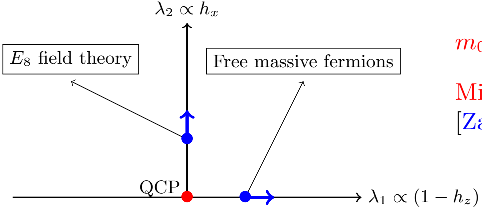
In our papers we studied the quench of $h_z > 1$ for $h_x = 0$ (Ising mass quench) and the quench of h_x for $h_z = 1$ (E_8 Toda mass quench).

3. Scaling Limit (Revisited)

- Throughout this course we have taken the scaling limit from spin chain models to IQFT a a given.
- This connection is important not only conceptually but also in a practical sense because any numerical tests of QFT predictions are generally done on discrete models in the scaling limit.
- For the model in the last slide, this scaling limit can be made a little more precise:

$$H = -J \sum_{j=1}^{N} \left[\sigma_j^x \sigma_{j+1}^x + h_x \sigma_j^x + h_z \sigma_j^z\right]$$

$$\mathcal{A} = \mathcal{A}_{\text{QCP}} - \lambda_1 \int dx dt\, \varepsilon(x,t) - \lambda_2 \int dx dt\, \sigma(x,t)$$



Ising Field Theory

$m_0 := 2J|1-h_z|, \quad h_x = 0, \quad v := 2Ja$

Minimal E_8 Toda Field Theory [Zamolodchikov'89]

$h_z = 1, \quad \lambda_2 \propto h_x, \quad m_0 := \kappa \lambda_2^{\frac{8}{15}}$

4. Global Mass Quench and Entanglement Dynamics

- Consider now a global mass quench $m_0 \mapsto m$.
- The entropy is defined as usual:

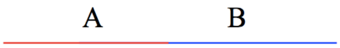
$$S_n(t) = \frac{\log(\mathrm{Tr}_A(\rho_A^n))}{1-n}$$

$$S(t) = \lim_{n\to 1} S_n(t)$$

$$\rho_A = \mathrm{Tr}_B(e^{-itH(m)}|0\rangle\langle 0|e^{itH(m)})$$

- $H(m)$ is the hamiltonian of the theory with mass m.
- $|0\rangle$ is the pre-quench ground state of $H(m_0)$.

- Consider the entanglement of two semi-infinite regions:



- As usual, the Rényi entropies can be expressed in terms of the one-point function of a BPTF:

$$\boxed{S_n(t) = \frac{\log(\varepsilon^{2\Delta_n}\,{}_n\langle 0|\mathcal{T}(0,t)|0\rangle_n)}{1-n}}$$

- $|0\rangle_n$ is the pre-quench ground state in the replica theory.
- Now this is a dynamical one-point function.

5. Quench Perturbation Theory

- We need a technique to compute dynamical one-point functions.
- One possibility is to generalize the perturbation theory proposed in [Delfino'14] to the BPTF.
- We start with an action

$$\mathcal{A} = \mathcal{A}_{\mathrm{QCP}} - \lambda \int dx dt \, \varphi(x,t)$$

  and consider a global quench whereby $\lambda \mapsto \lambda + \delta_\lambda$ at $t = 0$.
- δ_λ is the small perturbative parameter.
- Then, at first order, the post-quench state can be approximated by expanding $|\tilde{0}\rangle = S_{\delta_\lambda}|0\rangle$, where $S_{\delta_\lambda} = T \exp\left(-i\delta_\lambda \int dx dt \, \varphi(x,t)\right)$

$$|\tilde{0}\rangle = |0\rangle + \delta_\lambda \sum_{k=1}^{\infty} \sum_{a_k=1}^{N} \frac{2\pi}{k!} \int_{-\infty}^{\infty} \prod_{i=1}^{k} \frac{dp_i}{e_i(\theta_i)} \delta(P_{\mathrm{tot}}) \frac{F_k^{\varphi|a_1\ldots a_k}(\theta_1,\ldots,\theta_k)^*}{(2\pi)^k E_{\mathrm{tot}}} |\theta_1 \ldots \theta_k|0\rangle + \cdots$$

- where $E_{\mathrm{tot}} = \sum_i e_i(\theta_i)$ and $P_{\mathrm{tot}} = \sum_i p_i(\theta_i)$ are the total energy and momentum of the state.
- The momentum constraint connects to the quasi-particle picture.

6. Dynamic Expectation Values

- From this formula it is then relatively easy to find the expectation value of any local field.
- Defining $\delta\langle\mathcal{O}\rangle := \langle\tilde{0}|\mathcal{O}(0,t)|\tilde{0}\rangle - \langle\tilde{0}|\mathcal{O}(0,0)|\tilde{0}\rangle$ we have

$$\delta\langle\mathcal{O}\rangle = \delta_\lambda \sum_{k=1}^{\infty}\sum_{a_k=1}^{N}\frac{2\pi}{k!}\int_{-\infty}^{\infty}\prod_{i=1}^{k}\frac{dp_i}{2\pi e_i(\theta_i)}\frac{\delta(P_{\rm tot})}{E_{\rm tot}}$$

$$\times \mathrm{Re}\left(F_k^{\varphi|a_1\ldots a_k}(\theta_1,\ldots,\theta_k)^* F_k^{\mathcal{O}|a_1\ldots a_k}(\theta_1,\ldots,\theta_k)e^{-itE_{\rm tot}}\right) + \cdots$$

$$\delta\langle\mathcal{T}\rangle_n = n\,\delta_\lambda \sum_{k=1}^{\infty}\sum_{a_k=1}^{N}\frac{2\pi}{k!}\int_{-\infty}^{\infty}\prod_{i=1}^{k}\frac{dp_i}{2\pi e_i(\theta_i)}\frac{\delta(P_{\rm tot})}{E_{\rm tot}}$$

$$\times \mathrm{Re}\left(F_k^{\varphi|a_1\ldots a_k}(\theta_1,\ldots,\theta_k)^* F_k^{\mathcal{T}|a_1\ldots a_k}(\theta_1,\ldots,\theta_k)e^{-itE_{\rm tot}}\right) + \cdots$$

- Once more form factors are the building blocks!
- This is easily generalized to the branch point twist field. [OC-A, Lencsés, Szécsényi & Viti'19]
- In this case we only need form factors on one copy.

7. Dynamics of Entanglement

- The first few terms in the expansion read

$$\delta\langle\mathcal{T}\rangle_n = \delta_\lambda n \sum_{a=1}^{N} \frac{2}{m_{0,a}^2} F_a^{\varphi} F_a^{\mathcal{T}} \cos(m_a t) +$$

$$2\delta_\lambda n \sum_{a,b=1}^{N} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_a \mathrm{d}p_b}{2\pi e_a e_b} \frac{\delta(p_a+p_b)}{e_a+e_b} \mathrm{Re}\left[[F_{ab}^{\varphi}(\theta_{ab})]^* F_{ab}^{\mathcal{T}}(\theta_{ab}) e^{-i(\tilde{e}_a+\tilde{e}_b)t} \right] + \cdots + \mathcal{O}(\delta_\lambda^2)$$

$$\delta\langle\mathcal{T}\rangle_n = \underbrace{\delta_\lambda n \sum_{a=1}^{N} \frac{2}{m_{0,a}^2} F_a^{\varphi} F_a^{\mathcal{T}} \cos(m_a t) +}_{\text{Undamped Oscillations}}$$

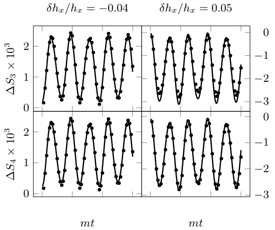
$$\underbrace{2\delta_\lambda n \sum_{a,b=1}^{N} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_a \mathrm{d}p_b}{2\pi e_a e_b} \frac{\delta(p_a+p_b)}{e_a+e_b} \mathrm{Re}\left[[F_{ab}^{\varphi}(\theta_{ab})]^* F_{ab}^{\mathcal{T}}(\theta_{ab}) e^{-i(\tilde{e}_a+\tilde{e}_b)t} \right]}_{\text{Damped Oscillations}} + \cdots$$

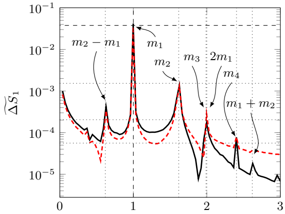
$$+ \underbrace{\mathcal{O}(\delta_\lambda^2)}_{\text{Linear Growth of EE}}$$

- This approach might lead us to miss the leading large-time behaviour of the entanglement entropy!
- But it gives us detailed knowledge of subleading oscillatory behaviours

8. Mass Quench in Minimal E_8 Toda Field Theory

- The Entanglement Entropy shows very slow growth and persistent undamped oscillations for medium-large times.
- These behaviours are not entirely surprising given existing results [Horvath, Kormos & Takacs'18], but they are surprisingly robust.





- The Fourier transform gives us a measurement of the one-particle form factors and the mass ratios.
- In this theory there are 8 particles with distinct masses m_a!
- Numerics performed with iTEBD.