

Lecture 3C: Composite Twist Fields and Field Counting

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1. Operator Identification

- Given a solution to the form factor equations, how do we know what local field $\mathcal{O}$ does it correspond to?
- There are several ways to do this: the constraints on form factor asymptotics [see page 9 Lecture 2A] will at least narrow down the possible solutions. Other factors are the spin and symmetries.
- One of the most useful checks is provided by the Δ-sum rule [Delfino, Simonetti & Cardy'96]

Δ-Sum Rule

$$\Delta_{\mathcal{O}} = -\frac{1}{2\langle\mathcal{O}\rangle}\int_0^\infty dr\, r\,\langle 0|\Theta(r)\mathcal{O}(0)|0\rangle$$

- Θ is the trace of the stress-energy tensor.
- We need to know the form factors of both fields. They must be non-zero for at least some of the same particle numbers.
- Then the Δ-sum rule gives us a direct connection between the form factors of a local field and the conformal dimension of its conformal (UV) counterpart [Zamolodchikov'89]

2. Identifying Branch Point Twist Fields

- In our original work [Cardy, OC-A & Doyon'08] we identified the form factors of the branch point twist field using the Δ-sum rule, symmetries (in the two models that we studied had $\mathbb{Z}_2$ symmetry meant that only even form factors were non-zero) and a certain idea of what the "simplest" solution should look like.
- In general though, there can be other twist fields associated with the same cyclic permutation symmetry (but they will have different conformal dimension). Their form factors will satisfy exactly the same form factor equations as $\mathcal{T}$.
- Today I want to discuss an example where the twist field form factor equations naturally give two possible solutions. I will talk about the massive Lee-Yang model, a massive perturbation of the non-unitary Lee-Yang minimal model with $c = -\frac{22}{5}$ [Lee & Yang'52; Cardy & Mussardo'89].

- The Lee-Yang minimal model is the simplest minimal model of CFT. It contains two primary fields $\{\mathbf{1}, \phi\}$. It is non-unitary in particular in the sense that $\Delta_\phi = -\frac{1}{5} < 0$.
- A massive perturbation of this theory (by the same field ϕ) gives the massive Lee-Yang model, with S-matrix [Cardy & Mussardo'89]

$$S(\theta) = \frac{\tanh \frac{1}{2}\left(\theta + \frac{2\pi i}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{2\pi i}{3}\right)}.$$

- Despite the non-unitary underlying CFT, this S-matrix is perfectly nice. There is a single particle in the spectrum and it can form a bound state with itself (hence the pole at $\theta = \frac{2\pi i}{3}$).
- This means that the two-particle form factor needs to have both kinematic poles and a bound state pole [see Lecture 2A].

- We can include the bound state pole in the (re)definition of the minimal form factor [see page 1 & 2 Lecture 2B]: [Bianchini, OC-A & Doyon'15]

$$F_{\mathbf{min}}^{\mathcal{T}}(\theta) = P(\theta) \exp\left[\int_0^\infty \frac{dt}{t} \frac{\sinh\frac{t}{3}\sinh\frac{t}{6}\cosh\left(\frac{t(n\pi+i\theta)}{\pi}\right)}{\sinh(nt)\cosh\frac{t}{2}}\right]$$

with $P(\theta) = \frac{\cosh\frac{\theta}{n}-1}{\cosh\frac{\theta}{n}-\cos\frac{2\pi}{3n}}$.

- Then, as before, we could write that

$$F_2^{\mathcal{T}}(\theta) = \frac{\langle\mathcal{T}\rangle\sin\frac{\pi}{n}}{2n\sinh\frac{i\pi+\theta}{2n}\sinh\frac{i\pi-\theta}{2n}} \frac{F_{\mathbf{min}}^{\mathcal{T}}(\theta)}{F_{\mathbf{min}}^{\mathcal{T}}(i\pi)}$$

is the solution to the form factor equations.

- However, this is not the most general solution that we can write (although it is the simplest and most natural for many theories)

$$F_2^{\mathcal{O}}(\theta) = \frac{\langle\mathcal{O}\rangle\sin\frac{\pi}{n}}{2n\sinh\frac{i\pi+\theta}{2n}\sinh\frac{i\pi-\theta}{2n}} \frac{F_{\mathbf{min}}^{\mathcal{O}}(\theta)}{F_{\mathbf{min}}^{\mathcal{O}}(i\pi)} + \kappa F_{\mathbf{min}}^{\mathcal{O}}(\theta)$$

5. Lee-Yang Form Factors

- Lee-Yang has no internal $\mathbb{Z}_2$ symmetry. This means that odd particle number form factors are generally non-vanishing.
- One way to fix the value of κ is to use the momentum space clustering decomposition property [see page 10 Lecture 2A]

$$\lim_{\theta\to\infty} F_2^{\mathcal{O}}(\theta) = \kappa = \frac{(F_1^{\mathcal{O}})^2}{\langle\mathcal{O}\rangle}.$$

and the bound state residue equation [see page 8 Lecture 2A]

$$-i \lim_{\theta\to\frac{2\pi i}{3}} (\theta - \frac{2\pi i}{3}) F_2^{\mathcal{O}}(\theta) = \Gamma F_1^{\mathcal{O}} \quad \text{with} \quad \Gamma = i\sqrt{2\sqrt{3}}.$$

- Combining these two equations we get a quadratic equation for $F_1^{\mathcal{O}}$ with solutions:

One-Particle Form Factors

$$F_1^{\mathcal{O}_\pm} = -\langle\mathcal{O}_\pm\rangle\Gamma \frac{\cos\frac{\pi}{3n} \pm 2\sin^2\frac{\pi}{6n}}{2n\sin\frac{\pi}{3n} f_\pm(\frac{2\pi i}{3}, n)} \quad \text{with} \quad f_\pm(\theta, n) := \frac{F_{\mathbf{min}}^{\mathcal{O}_\pm}(\theta)}{P(\theta)}$$

6. Identifying Branch Point Twist Fields

- Looking again at these solutions, the question is what kind of branch point twist fields are the fields $\mathcal{O}_\pm$?

One-Particle Form Factors

$$F_1^{\mathcal{O}_\pm} = -\langle \mathcal{O}_\pm \rangle \Gamma \frac{\cos\frac{\pi}{3n} \pm 2\sin^2\frac{\pi}{6n}}{2n\sin\frac{\pi}{3n} f_\pm(\frac{2\pi i}{3}, n)} \quad \text{with} \quad f_\pm(\theta, n) := \frac{F_{\mathbf{min}}^{\mathcal{O}_\pm}(\theta)}{P(\theta)}$$

- There are some clues we can get without much work. We know that $F_1^{\mathcal{T}} = 0$ at $n = 1$. We see that this holds for $\mathcal{O}_-$!
- For $\mathcal{O}_+$ the form factor is not zero at $n = 1$. It is:

$$\lim_{n\to 1} \frac{F_1^{\mathcal{O}_+}}{\langle \mathcal{O}_+ \rangle} = \frac{F_1^{\phi}}{\langle \phi \rangle} = \frac{i\sqrt{2}}{\sqrt[4]{3} f(\frac{2\pi i}{3}, 1)}$$

  this was computed in [Zamolodchikov'91]
- This strongly suggests that $\mathcal{O}_- = \mathcal{T}$ and $\mathcal{O}_+ =: \mathcal{T}\phi :$.

7. Composite Branch Point Twist Fields

- The field $\mathcal{O}_+ =: \mathcal{T}\phi :$ is an example of a composite branch point twist field. Recall that we saw the conformal dimensions of these fields in Lecture 1B:

Branch Point Twist Field Conformal Dimension(s)

$$\Delta_{\mathcal{T}} = \frac{c}{24}\left(n - \frac{1}{n}\right) \qquad \Delta_{:\mathcal{T}\phi:} = \frac{c_{\text{eff}}}{24}\left(n - \frac{1}{n}\right)$$

[Knizhnik'87; Kac & Wakimoto'90; Bouwknegt'96; Borisov et al.'98]

- Where $c_{\text{eff}} = c - 24\Delta_\phi$. In this case $c, \Delta_\phi < 0$ but $c_{\text{eff}} > 0$ so the field $:\mathcal{T}\phi:$ is the BPTF with the least positive dimension.

- We can make sense of this field as corresponding to the leading field in the conformal OPE of $\mathcal{T}$ and ϕ in a replica CFT:.

$$:\mathcal{T}\phi := n^{2\Delta - 1} \lim_{x\to y} |x-y|^{2\Delta_\phi\left(1-\frac{1}{n}\right)} \sum_{j=1}^{n} \mathcal{T}(x)\phi_j(y)$$

ϕ_j is the field ϕ in copy j and the normalization ensures conformal normalization of the two point function of $:\mathcal{T}\phi:$.

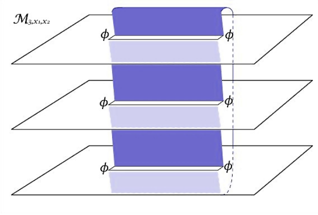
8. Composite BPTFs and Entanglement

- This kind of field appeared in the context of entanglement in [OC-A, Doyon & Levi'11; Levi'12]
- Later on we proposed a measure of entanglement for non-unitary CFT where, for instance:

$$ {}_n\langle 0|\mathcal{T}(x_1)\mathcal{T}^\dagger(x_2)|0\rangle_n \quad \mapsto \quad \frac{{}_n\langle 0| :\mathcal{T}\phi: (x_1) :\mathcal{T}\phi:^\dagger (x_2)|0\rangle_n}{\langle 0|\phi(x_1)\phi(x_2)|0\rangle^n} $$

  we showed that this mapping gives an entanglement entropy where all formulae hold as before with the replacement $c \mapsto c_{\text{eff}}$. [Bianchini, OC-A, Doyon, Levi & Ravanini'15]

- There is some evidence in lattice models [Bianchini & Ravanini'16; Couvreur, Jacobsen & Saleur'16] that this works, but also some people that disagree [Dupic, Estienne & Ikhlef'17] ☹



9. Composite Twist Fields More Generally

- This example gives an idea of how the form factor programme connects to the identification of the operator content of IQFT.
- This is not a new problem and there are many beautiful papers where this is addressed, for example [Cardy & Mussardo'90] for the Ising model, and [Koubek & Mussardo'93] for sinh-Gordon or even [OC-A & Fring'01] for the Homogeneous sine-Gordon model.
- An interesting extension of some of these ideas is to consider fields $:\mathcal{T}\phi:$ where ϕ is also a twist field. For instance in the Ising model, one could look at $:\mathcal{T}\sigma:$ and ask what are its form factors.
- If ϕ is a twist field, then $:\mathcal{T}\phi:$ will satisfy a new set of form factor equations that incorporate both symmetries. These were written for the first time in [Horváth & Calabrese'20] and the fields have been named Symmetry Resolved Twist Fields.
- A new measure of entanglement, the symmetry resolved entanglement [Goldstein & Sela'18] can be computed using these fields.


THE
STORY
GOES ON
for branch point twist fields....
WATCH THIS SPACE!