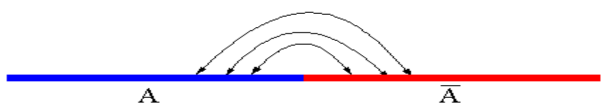
Entanglement Measures: Exercises

1. Consider the bi-partite entanglement entropy of a single interval, defined as

$$S_A = -\mathbf{Tr}(\rho_A \log \rho_A).$$

   Show that $S_A = S_{\bar{A}}$ where $\bar{A}$ is the complement of A.



   This suggests that entanglement may be (in some situations) regarded as counting the number of "links" between the two sub-systems.

2. Show that the maximal entanglement entropy of a two spin system with spin s is $\log(2s+1)$. Hint: You may show this by proving that a maximum is achieved when all eigenvalues of the density matrix are equal (this is the same property as for Shanon's entropy in classical information theory). You can think of this as an optimization problem with one Lagrange multiplier as the sum of the eigenvalues of the density matrix is fixed to 1.

3. Given that the entropy of region of length ℓ with two boundary points in an infinite unitary critical system diverges as:

$$S(\ell) = \frac{c}{3} \log \frac{\ell}{\varepsilon},$$

   where ε is a non-universal short-distance cut-off, use conformal maps to find the equivalent expressions for the entanglement entropy of sub-system of length ℓ with two boundary points within a finite system of length L. Hint: You just need to find the appropriate conformal map.

4. The branch point twist field and its conjugate are primary fields in a replica conformal field theory of central charge nc, where c is the central charge of each replica. Compute the expectation value of the stress-energy tensor of the replica CFT in the n-sheeted manifold seen in Lecture 1A, that is $\langle T(w) \rangle_{\mathcal{M}_{n,x_1,x_2}}$, where x_1, x_2 are the branch points. Hint: Again you need to find the appropriate conformal map and use the transformation law of the stress-energy tensor in CFT.

5. We have seen that the replica partition function of a finite connected interval is proportional to the two-point function of branch point twist fields $\langle \mathcal{T}(x_1)\mathcal{T}^\dagger(x_2) \rangle = |x_1 - x_2|^{-4\Delta_\mathcal{T}}$. From this and the result of the previous question, find the conformal dimension of the branch point twist fields $\Delta_\mathcal{T} = \frac{c}{24}\left(n - \frac{1}{n}\right)$.

6. Consider the following CFT results for two-, three- and four-point functions of primary fields:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = |x_1 - x_2|^{-4\Delta}\,,$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\mathcal{C}_{123}}{|x_1-x_2|^{2\Delta_1+2\Delta_2-2\Delta_3}|x_2-x_3|^{2\Delta_2+2\Delta_3-2\Delta_1}|x_1-x_3|^{2\Delta_1+2\Delta_3-2\Delta_2}}\,,$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \mathcal{F}(x)\left[\frac{|x_1-x_3||x_2-x_4|}{|x_1-x_2||x_3-x_4||x_1-x_4||x_2-x_3|}\right]^{4\Delta}$$

the last equation is true when the dimensions of all four fields are the same Δ and $\mathcal{F}(x)$ is a model dependent function of the cross-ratio

$$x = \frac{|x_1-x_2||x_3-x_4|}{|x_1-x_3||x_2-x_4|}.$$

Use the first two formulae to obtain the formulae for the Rényi and von Neumann entropies of a finite interval and for the (replica) logarithmic negativity of adjacent regions. For free fermions the function $\mathcal{F}(x) = 1$. In this case, find the Rényi and entanglement entropies of two disconnected regions.