Measures of Entanglement: Solutions to Exercises

1. **Entanglement entropy is not directional**
   Let $|\psi\rangle = |A\rangle \otimes |\bar{A}\rangle$ be an eigenstate of a quantum many body system. Let f and $\bar{f}$ be anti-linear maps defined as:
$$f : \mathcal{A} \to \bar{\mathcal{A}}$$
$$f|A\rangle = \langle A|\psi\rangle$$
   and
$$\bar{f} : \bar{\mathcal{A}} \to \mathcal{A}$$
$$\bar{f}|\bar{A}\rangle = \langle \bar{A}|\psi\rangle$$
   Then the reduced density matrices can be defined as
$$\rho_A = \bar{f}f \qquad \textbf{and} \qquad \rho_{\bar{A}} = f\bar{f}$$
   thus, they have the same set of non-zero eigenvalues (same degeneracies).

2. **Maximization of Shanon's entropy**
   Since it is a two spin system and we are tracing out over one of the spins, the dimension of the density matrix is $2s+1 \times 2s+1$. There are therefore $2s+1$ positive eigenvalues λ_i with $i = 1, \ldots, 2s+1$. The entanglement entropy is then defined as
$$S_A = -\sum_{i=1}^{2s+1} \lambda_i \log \lambda_i.$$
   and we want to find the maximum of this quantity subject to the constraint that $\sum_{i=1}^{2s+1} \lambda_i = 1$. This is a typical optimization problem which we can solve by defining the function
$$f(\lambda_1, \ldots, \lambda_{2s+1}) = -\sum_{i=1}^{2s+1} \lambda_i \log \lambda_i + \mu(\sum_{i=1}^{2s+1} \lambda_i - 1),$$
   and solving the equations
$$\frac{\partial f}{\partial \lambda_i} = 0 \quad \textbf{for} \quad i = 1, \ldots, 2s+1 \qquad \textbf{and} \qquad \frac{\partial f}{\partial \mu} = 0.$$
   The first $2s+1$ equations become
$$-\log \lambda_i - 1 + \mu = 0 \quad \textbf{for} \quad i = 1, \ldots, 2s+1$$
   that is $\lambda_i = e^{\mu-1}$ for every value of i. Inserting this result into the constraint we have that
$$\sum_{i=1}^{2s+1} \lambda_i = (2s+1)e^{\mu-1} = 1,$$

that is

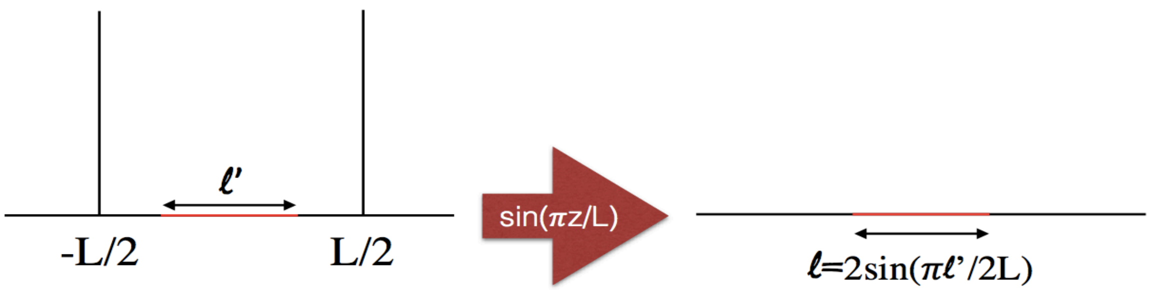
$$e^{\mu-1} = \frac{1}{2s+1} = \lambda_i \quad \textbf{for} \quad i = 1, \ldots, 2s+1.$$

This gives the value anticipated before

$$S_A = \log(2s+1).$$

In particular, the Bell state that was shown as an example in the lecture has entanglement entropy $\log 2$ which maximal for any two-spin system with spin $\frac{1}{2}$.

3. **Entanglement Entropy of Finite Systems**
Consider the configuration below:



The map $f(z) = \sin\frac{\pi z}{L}$ maps the points $\pm L/2$ into the points ± 1. In fact, it maps the vertical strip on the l.h.s. into the upper half plane and maps its boundaries onto the real line. This is therefore the map that we need to use. So we expect a formula of the type

$$S(\ell, L) = \frac{c}{3}\log\left(\sin\frac{\pi\ell}{L}\right) + \text{constants}$$

We would like however, to ensure as well that when $L \to \infty$ we recover the original result. In order to do this we may express the constants in a more convenient way as:

$$S(\ell, L) = \frac{c}{3}\log\left(\frac{L}{\pi\varepsilon}\sin\frac{\pi\ell}{L}\right).$$

This result appeared in C. Holzhey, F. Larsen and F. Wilczek, Nucl. Phys. B424 (1994) 443-467 but it is best known in our community from the work of P. Calabrese and J.L. Cardy, J. Stat. Mech. 0406:P06002 (2004). It is an important formula specially for numerics as in most cases only finite discrete systems can be simulated on a computer, so this is the kind of scaling of EE that is most frequently observed in numerics.

4.

5. **Branch Point Twist Fields and Multi-Sheeted Riemann Surfaces**
The solution to problems 4 and 5 can be found in our paper J.L. Cardy, OC-A and B. Doyon, J. Stat. Phys. 130 129-168 (2008), in section 2.1. There is a similar derivation in P. Calabrese and J.L. Cardy, J. Stat. Mech. 0406:P06002 (2004) but the definition of their field is slightly different and the conformal dimension that they get is scaled by a factor n.

6. **Entanglement Formulae from CFT Correlators**

Entanglement Entropy of one Interval: Let

$$\langle \mathcal{T}(0)\mathcal{T}^\dagger(\ell)\rangle = \frac{1}{\ell^{4\Delta_\mathcal{T}}}$$

then

$$\log(\varepsilon^{4\Delta_\mathcal{T}}\langle \mathcal{T}(0)\mathcal{T}^\dagger(\ell)\rangle) = 4\Delta_\mathcal{T}\log\frac{\varepsilon}{\ell} \qquad \Rightarrow \qquad S_n(\ell) = \frac{4\Delta_\mathcal{T}}{1-n}\log\frac{\varepsilon}{\ell} = \frac{(1+n)c}{6n}\log\frac{\ell}{\varepsilon}$$

thus, the von Neumann entropy is

$$S(\ell) = \lim_{n\to 1}\frac{(1+n)c}{6n}\log\frac{\ell}{\varepsilon} = \frac{c}{3}\log\frac{\ell}{\varepsilon}\,.$$

Therefore the results for the entropies of one connected interval are a direct consequence of the structure of two-point functions of primary fields in CFT.

Replica Negativity and Entanglement of two Disconnected Regions in CFT:

Let us now look at the logarithmic negativity and its replica version. Here the starting point was the two point function

$$\langle \mathcal{T}(x_1)\mathcal{T}^\dagger(y_1)\mathcal{T}^\dagger(x_2)\mathcal{T}(y_2)\rangle$$

If we were to consider this four-point function directly, then we could use the four-point function formula given in the exercise. This would give us

$$\mathcal{E}_n(x_1,y_1,x_2,y_2) = \log(\varepsilon^{8\Delta_\mathcal{T}}\langle \mathcal{T}(x_1)\mathcal{T}^\dagger(y_1)\mathcal{T}^\dagger(x_2)\mathcal{T}(y_2)\rangle)$$

$$= 4\Delta_\mathcal{T}\log\frac{|x_1-x_2|}{\varepsilon} + 4\Delta_\mathcal{T}\log\frac{|y_1-y_2|}{\varepsilon} - 4\Delta_\mathcal{T}\sum_{i,j}\log\frac{|x_i-y_j|}{\varepsilon} + \log\mathcal{F}(x)$$

and for the Rényi entropies of two disconnected regions the relevant four-point function was

$$\langle \mathcal{T}(x_1)\mathcal{T}^\dagger(y_1)\mathcal{T}(x_2)\mathcal{T}^\dagger(y_2)\rangle$$

we get the same but with a different function $\tilde{\mathcal{F}}(x)$ as the correlator is different

$$S_n(x_1,x_2,x_3,x_4) = \frac{4\Delta_\mathcal{T}}{1-n}\log\frac{|x_1-x_2|}{\varepsilon} + \frac{4\Delta_\mathcal{T}}{1-n}\log\frac{|y_1-y_2|}{\varepsilon} - \frac{4\Delta_\mathcal{T}}{1-n}\sum_{i,j}\log\frac{|x_i-y_j|}{\varepsilon} + \frac{\log\tilde{\mathcal{F}}(x)}{1-n}.$$

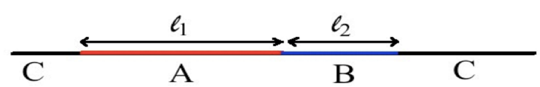
If $\tilde{\mathcal{F}}(x) = 1$ and $c = \frac{1}{2}$ and we take the limit $n \to 1$ we get the formula for the entanglement entropy of disconnected regions in a free fermion theory

$$S(x_1, x_2, x_3, x_4) = \frac{c}{3} \sum_{i,j} \log \frac{|x_i - y_j|}{\varepsilon} - \frac{c}{3} \log \frac{|x_1 - x_2|}{\varepsilon} - \frac{c}{3} \log \frac{|y_1 - y_2|}{\varepsilon} \,.$$

found in P. Calabrese and J.L. Cardy, J. Stat. Mech. 0406:P06002 (2004). For free fermions, this result generalizes in an obvious way to multiple disconnected regions (just have to increase the range of indices i, j). In the original paper it was thought the result held more widely in CFT. It was later shown by Furukawa, Pasquier and Shiraishi, Phys. Rev. Lett. 102, 170602 (2009) that for generic CFT this behaviour is corrected by the function $\tilde{\mathcal{F}}(x)$ in a non-negligible way.

In a series of papers (starting 2010) Calabrese, Cardy and Tonni (and later also with other collaborators) then studied these two four-point functions in great detail for the compactified free boson. They chose this theory mainly because the orbifold partition functions had been computed using vertex operators in L. Dixon, D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B282, (1987) 13-73.

Replica Negativity of Adjacent Regions in CFT



This is the most interesting case where one can get non-trivial results. It corresponds to the limit when the distance between regions A and B goes to zero. From the four-point function viewpoint, this amounts to bringing the two fields $\mathcal{T}^\dagger$ very close to each other. The resulting three-point function was studied for the first time in the papers P. Calabrese, J.L. Cardy and E. Tonni, Phys. Rev. Lett. 109, 130502 (2012); J. Stat. Mech. (2013) P02008. The discussion below follows exactly from the beautiful ideas presented in those papers.

Let us examine the OPE of two branch point twist fields $\mathcal{T}$

$$\mathcal{T}(x_1)\mathcal{T}(x_2) = \mathcal{C}_{\mathcal{T}\mathcal{T}}^{\mathcal{O}} |x_1 - x_2|^{2\Delta_{\mathcal{O}} - 4\Delta_{\mathcal{T}}} \mathcal{O} + \cdots$$

where $\mathcal{O}$ is the leading field in the expansion. The expansion must respect the cyclic permutation symmetry which means that the field $\mathcal{O}$ must also be a branch point twist field implementing the double action of $\mathcal{T}$. This double action amounts to going from copy j to copy $j+2$ so it is natural to call this field $\mathcal{O} = \mathcal{T}^2$. A key idea from the papers above is the realization that the field $\mathcal{T}^2$ is markedly different depending on whether n is even or odd. It is easy to reason that if n is odd, repeated action of $\mathcal{T}^2$ starting from a given copy would allow us to visit every other copy. In other words, for n odd, the field $\mathcal{T}^2$ is exactly like the field $\mathcal{T}$ up to relabelling of the copies. In particular then $\Delta_{\mathcal{T}^2} = \Delta_{\mathcal{T}}$ for $n$odd. In this case, the replica negativity becomes just

$$\mathcal{E}_{n_o}(\ell_1, \ell_2) = \log(\varepsilon^{6\Delta_{\mathcal{T}}} \langle \mathcal{T}(-\ell_1)\mathcal{T}^\dagger(0)\mathcal{T}(\ell_2) \rangle)$$

and using the formula for three-point functions this gives

$$\mathcal{E}_{n_o}(\ell_1,\ell_2) = \log\left(\frac{\varepsilon^{6\Delta_\mathcal{T}}\mathcal{C}_{\mathcal{T}\mathcal{T}^\dagger\mathcal{T}}}{(\ell_1\ell_2(\ell_1+\ell_2))^{2\Delta_\mathcal{T}}}\right) = 2\Delta_\mathcal{T}\left(\log\frac{\ell_1}{\varepsilon}+\log\frac{\ell_2}{\varepsilon}+\log\frac{\ell_1+\ell_2}{\varepsilon}\right)+\log\mathcal{C}_{\mathcal{T}\mathcal{T}^\dagger\mathcal{T}}\,.$$

Except for the fusion constant $\mathcal{C}_{\mathcal{T}\mathcal{T}^\dagger\mathcal{T}}$, the rest of the formula tends to zero when $n_o \to 1$.

The more interesting and meaningful result is obtained when n is even and indeed it is from n even that we can then obtain the logarithmic negativity. For n even the field $\mathcal{T}^2$ is also associated with cyclic permutation symmetry. However, in this case the copies decouple into two independent sets. If we start at an even copy $\mathcal{T}^2$ will only allow us to visit other even copies and similarly if start with an odd one. Thus, for n even we can formally write that

$$\mathcal{T}^2 = \mathcal{T}_{\frac{n_e}{2}} \otimes \mathcal{T}_{\frac{n_e}{2}}\,,$$

where $\mathcal{T}_{\frac{n_e}{2}}$ is a standard branch point twist field in a $\frac{n_e}{2}$ copy theory. Therefore its conformal dimension is $\Delta_{\mathcal{T}^2} = \frac{c}{12}\left(\frac{n_e}{2}-\frac{2}{n_e}\right)$. This means that for n even, the replica logarithmic negativity of CFT can be written as

$$\mathcal{E}_{n_e}(\ell_1,\ell_2) = \log(\varepsilon^{4\Delta_\mathcal{T}-2\Delta_{\mathcal{T}^2}}\langle\mathcal{T}(-\ell_1)(\mathcal{T}^\dagger)^2(0)\mathcal{T}(\ell_2)\rangle)$$

and from the three-point function formula it follows that

$$\mathcal{E}_{n_e}(\ell_1,\ell_2) = \log\left(\frac{\varepsilon^{4\Delta_\mathcal{T}+2\Delta_{\mathcal{T}^2}}\mathcal{C}_{\mathcal{T}(\mathcal{T}^\dagger)^2\mathcal{T}}(\ell_1\ell_2)^{-2\Delta_{\mathcal{T}^2}}}{(\ell_1+\ell_2)^{4\Delta_\mathcal{T}-2\Delta_{\mathcal{T}^2}}}\right)$$

$$= -2\Delta_{\mathcal{T}^2}\log\frac{\ell_1\ell_2}{\varepsilon(\ell_1+\ell_2)} - 4\Delta_\mathcal{T}\log\frac{\ell_1+\ell_2}{\varepsilon} + \log\mathcal{C}_{\mathcal{T}(\mathcal{T}^\dagger)^2\mathcal{T}}\,,$$

and therefore the logarithmic negativity of adjacent regions is

$$\lim_{n_e\to 1}\mathcal{E}_{n_e}(\ell_1,\ell_2) = \frac{c}{4}\log\frac{\ell_1\ell_2}{\varepsilon(\ell_1+\ell_2)} + \mathcal{E}_\perp\,,$$

where

$$\mathcal{E}_\perp = \lim_{n_e\to 1}\mathcal{C}_{\mathcal{T}(\mathcal{T}^\dagger)^2\mathcal{T}}\,.$$