Twist Field Form Factors: Solutions to Exercises

1. **Minimal Form Factors**
   This is a very simple exercise. The first identity $F_{\mathbf{min}}^{\mathcal{T}|ab}(\theta) = S_{ab}(\theta)F_{\mathbf{min}}^{\mathcal{T}|ba}(-\theta)$ holds if

$$\frac{f_{ab}(t)}{\sinh(nt)}\sin^2\left(\frac{it}{2}\left(n+\frac{i\theta}{\pi}\right)\right) = f_{ab}(t)\sinh\frac{t\theta}{\pi} + \frac{f_{ab}(t)}{\sinh(nt)}\sin^2\left(\frac{it}{2}\left(n-\frac{i\theta}{\pi}\right)\right)$$

$$= \frac{f_{ab}(t)}{\sinh(nt)}\left(\sinh(nt)\sinh\frac{t\theta}{\pi} + \sin^2\left(\frac{it}{2}\left(n-\frac{i\theta}{\pi}\right)\right)\right)$$

   Since $\sin^2(i(x+y)) = \sin^2(i(x-y)) - \sinh(2x)\sinh(2y)$ we have that

$$\sin^2\left(\frac{it}{2}\left(n-\frac{i\theta}{\pi}\right)\right) = \sin^2\left(\frac{it}{2}\left(n+\frac{i\theta}{\pi}\right)\right) - \sinh\left(nt\right)\sinh\frac{t\theta}{\pi}$$

   which proves the identity. Similarly, to prove that $F_{\mathbf{min}}^{\mathcal{T}|ab}(\theta) = F_{\mathbf{min}}^{\mathcal{T}|ab}(-\theta+2\pi in)$ we need to show that

$$\sin^2\left(\frac{it}{2}\left(n+\frac{i\theta}{\pi}\right)\right) = \sin^2\left(\frac{it}{2}\left(n+\frac{i(-\theta+2\pi in)}{\pi}\right)\right) = \sin^2\left(\frac{it}{2}\left(-n-\frac{i\theta}{\pi}\right)\right),$$

   so it also holds. For $n=1$ this also gives us the construction procedure for the minimal form factors of more standard local fields. Here we assumed that $f_{ab}(\theta) = f_{ba}(\theta)$ which is equivalent to parity invariance of the S-matrix. This is not true for all theories but for most.

2. **Two Particle Form Factor**
   To check Watson's equations we need to check that $F_2^{\mathcal{T}|(a,j)(b,j)}(\theta) := g_{ab}(\theta)$ satisfies:

$$g_{ab}(\theta) = S_{ab}(\theta)g_{ba}(-\theta) = g_{ab}(-\theta+2\pi in).$$

   The equation $g_{ab}(\theta) = g_{ab}(-\theta+2\pi in)$ holds by construction because it is satisfied by the minimal form factor (by definition) and the rest of the formula involves a function which is invariant under $2\pi in$ shifts of θ. The equation $g_{ab}(\theta) = S_{ab}(\theta)g_{ba}(-\theta)$ holds also by construction because the minimal form factor satisfies it and the rest of the function is even in θ.

   To check the kinematic residue equation we need to verify that

$$\lim_{\theta\to i\pi}(\theta - i\pi)g_{ab}(\theta) = iF_0^{\mathcal{T}} = i\langle\mathcal{T}\rangle.$$

   This can be easily checked directly from the formula.

3. **Form Factor Relations**
The identity:
$$F_2^{\mathcal{T}|(a,i)(b,j)}(\theta) = F_2^{\mathcal{T}^\dagger|(a,n-i)(b,n-j)}(\theta),$$
is a consequence of the definition of the two twist fields. They implement the opposite cyclic permutation symmetries.

The formula,
$$F_2^{\mathcal{T}|(a,i)(b,i+k)}(\theta) = F_2^{\mathcal{T}|(a,j)(b,j+k)}(\theta),$$
follows from the fact that all copies are identical, so form factors where all the copy numbers are just shifted by a common value must be identical.

The relation,
$$F_2^{\mathcal{T}|(a,1)(b,j)}(\theta) = F_2^{\mathcal{T}|(b,1)(a,1)}(2\pi(j-1)i - \theta),$$
is a consequence of Watson's equations. If $j = 1$ the equation does not hold so we must have $j > 1$. By crossing we have
$$F_2^{\mathcal{T}|(a,1)(b,j)}(\theta) = F_2^{\mathcal{T}|(b,j-1)(a,1)}(-\theta + 2\pi i),$$
Using the first Watson equation and assuming $j - 1 \neq 1$ (if $j = 2$ the proof is finished!) we can write that
$$F_2^{\mathcal{T}|(b,j-1)(a,1)}(-\theta + 2\pi i) = F_2^{\mathcal{T}|(a,1)(b,j-1)}(\theta - 2\pi i)$$
because the S-matrix is that of n disconnected copies so $S_{(a,i)(b,j)}(\theta) = (S_{ab}(\theta))^{\delta_{ij}}$. Repeating the same steps once more we find that
$$F_2^{\mathcal{T}|(a,1)(b,j-1)}(\theta - 2\pi i) = F_2^{\mathcal{T}|(b,j-2)(a,1)}(-\theta + 4\pi i) = F_2^{\mathcal{T}|(a,1)(b,j-2)}(\theta - 4\pi i).$$
Eventually, after $j - 1$ iterations we will find:
$$F_2^{\mathcal{T}|(a,1)(b,j)}(\theta) = F_2^{\mathcal{T}|(b,1)(a,1)}(-\theta + 2\pi i(j-1)),$$
as we wanted to prove.

The fourth relation follows from the previous ones:
$$F_2^{\mathcal{T}^\dagger|(a,1)(b,j)}(\theta) = F_2^{\mathcal{T}|(a,n-1)(b,n-j)}(\theta) = F_2^{\mathcal{T}|(a,1)(b,n+2-j)}(\theta) = F_2^{\mathcal{T}|(b,1)(a,1)}(-\theta+2\pi i(n+2-j-1)),$$
$$= F_2^{\mathcal{T}|(b,1)(a,1)}(-\theta + 2\pi i(1-j) + 2\pi i n) = F_2^{\mathcal{T}|(a,1)(b,1)}(\theta + 2\pi i(j-1)),$$
the third identity follows from the invariance under global shift of copies. So we shift both copies by 2 and then use the property that $n + j \equiv j$ because of cyclic permutation symmetry. The fifth relation follows again from Watson's equations ($2\pi i n$-shift property).

The final relation follows from the same properties now generalized to a larger number of particles. For instance consider the three-particle form factor in a theory with $n = 3$:
$$F^{\mathcal{T}|(a,3)(b,2)(c,1)}(\theta_1, \theta_2, \theta_3) = F^{\mathcal{T}|(c,1)(b,2)(a,3)}(\theta_3, \theta_2, \theta_1) = F^{\mathcal{T}|(b,2)(a,3)(c,2)}(\theta_2, \theta_1, \theta_3 - 2\pi i)$$

$$= F^{\mathcal{T}|(b,2)(c,2)(a,3)}(\theta_2, \theta_3 - 2\pi i, \theta_1) = F^{\mathcal{T}|(c,2)(a,3)(b,3)}(\theta_3 - 2\pi i, \theta_1, \theta_2 - 2\pi i)$$
$$= F^{\mathcal{T}|(a,3)(b,3)(c,3)}(\theta_1, \theta_2 - 2\pi i, \theta_3 - 4\pi i) = F^{\mathcal{T}|abc}(\theta_1, \theta_2 - 2\pi i, \theta_3 - 4\pi i)\,.$$

All the identities follow from Watson's equations. This looks a bit different from the formula I wrote but recall that the form factor only depends on rapidity differences so one can add a constant to all rapidities without changing the form factor. This implies that

$$F^{\mathcal{T}|abc}(\theta_1, \theta_2 - 2\pi i, \theta_3 - 4\pi i) = F^{\mathcal{T}|abc}(\theta_1 + 4\pi i, \theta_2 + 2\pi i, \theta_3),$$

which is the formula given in my lecture.

4. **Analytic Continuation through the Cotangent Trick**
We have that

$$h(\theta, n) = \sum_{i,j=1}^{n} h_{ij}(\theta, n).$$

Because all copies are identical we have that

$$h(\theta, n) = \sum_{i,j=1}^{n} h_{ij}(\theta, n) = n \sum_{j=1}^{n} h_{1j}(\theta, n),$$

and due to the monodromy properties of the form factors, every two-particle form factor can be related to $h_{11}(\theta, n)$ by employing Watson's equations:

$$\begin{aligned} n\sum_{j=1}^{n} h_{1j}(\theta, n) &= n(h_{11}(\theta, n) - h_{11}(-\theta, n)) + n\sum_{j=1}^{n} h_{11}(-\theta + 2\pi i(j-1), n) \\ &= n(h_{11}(\theta, n) - h_{11}(-\theta, n)) + n\sum_{j=0}^{n-1} h_{11}(-\theta + 2\pi i j, n). \end{aligned}$$

In order to compute the derivative w.r.t. n of such a sum we need to get rid of the sum in n. This may be achieved through the well-known cotangent trick which we also used in the paper J. L. Cardy, O. A. Castro-Alvaredo and B. Doyon, J. Stat. Phys. (2008) 130 129-168. The idea is to replace the sum by an integral of the form:

$$\frac{1}{2\pi i}\oint \pi\cot(\pi z) s(\theta, z) dz,$$

where $s(\theta, z) = h_{11}(-\theta + 2\pi i z, n)$ and the integration contour is a rectangle in the complex plane with corners at $(iL - \epsilon, -iL - \epsilon, n + iL - \epsilon, n - iL - \epsilon)$ with $\epsilon \ll 1$. The ϵ ensures that the $j = 0$ term in the sum is included. The generic form of the two-particle form factors is

$$h_{11}(\theta, n) = \frac{\langle \mathcal{T} \rangle \sin\frac{\pi}{n}}{2n \sinh\frac{i\pi + \theta}{2n} \sinh\frac{i\pi - \theta}{2n}} \frac{f(\theta, n)}{f(i\pi, n)},$$

where $f(\theta)$ is a model-dependent minimal form factor (assume that we look at a theory with a single particle type, such a free Boson or a free Fermion). As we can see the form factor has kinematic poles at $\theta = i\pi$ and $\theta = i\pi(2n-1)$. It is also clear that $s(\theta, z)$ decays exponentially as $\Im z \to \pm\infty$ and the contributions from the horizontal segments in the rectangle will therefore vanish as $L \to \infty$. Thanks to the quasi-periodicity of the integrand, $s(\theta, z+n) = s(-\theta, -z) = S(\theta - 2\pi i z)s(\theta, z)$, where $S(\theta)$ is the two-particle scattering matrix, the contributions from the vertical pieces amounts to

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(S(\theta-2\pi i(z-\epsilon))-1)\pi\cot(\pi(z-\epsilon))s(\theta,z-\epsilon)dz$$

This will give a non-vanishing contribution for all theories (except the free Boson).

The sum of the residues of the poles of the cotangent function inside the contour reproduces the original sum. However, there are additional poles at $z = \frac{1}{2} + \frac{\theta}{2\pi i}$ and $z = n - \frac{1}{2} + \frac{\theta}{2\pi i}$ inside the contour which correspond to the kinematic poles of the form factor. The sum of the residues associated with these poles gives:

$$i\pi\cot\left(\frac{\pi}{2}+\frac{\theta}{2i}\right) - i\pi\cot\left(\pi n - \frac{\pi}{2}+\frac{\theta}{2i}\right) = 0.$$

The sum above is zero for n integer as the cotangent function is π periodic. This means that the integral (4) is in fact the only non-vanishing contribution in this case and it is obviously model-dependent.

Therefore the analytically-continued sum $\tilde{f}(\theta, n)$ is given by

$$\begin{aligned}
&\tilde{h}(\theta,n) = n(h_{11}(\theta,n)-h_{11}(-\theta,n))\\
&+\lim_{\epsilon\to 0}\frac{n}{2\pi i}\int_{-i\infty}^{i\infty}(S(\theta-2\pi i(z-\epsilon))-1)\pi\cot(\pi(z-\epsilon))s(\theta,z-\epsilon)dz\\
&= n(h_{11}(\theta,n)-h_{11}(-\theta,n))\\
&-\lim_{\epsilon\to 0}\frac{n}{4\pi i}\int_{-\infty}^{\infty}(S(\theta-\beta+i\epsilon)-1)\coth\frac{\beta-i\epsilon}{2}h_{11}(-\theta+\beta-i\epsilon,n)d\beta, \qquad (1)
\end{aligned}$$

where we changed variables to $\beta = 2\pi i z$ and replaced $2\pi\epsilon \to \epsilon$. We can now differentiate w.r.t. n and take the $n \to 1$ limit. The result depends on the minimal form factor, so it is model-dependent. Note that $h_{11}(\theta, 1) = 0$ by construction, so some of the terms are directly vanishing. The non-trivial contributions come from

$$\begin{aligned}
&\frac{\partial}{\partial n}\tilde{h}(\theta,n) = n\frac{\partial}{\partial n}(h_{11}(\theta,n)-h_{11}(-\theta,n))\\
&-\lim_{\epsilon\to 0}\frac{n}{4\pi i}\int_{-\infty}^{\infty}(S(\theta-\beta+i\epsilon)-1)\coth\frac{\beta-i\epsilon}{2}\frac{\partial}{\partial n}h_{11}(-\theta+\beta-i\epsilon,n)d\beta.
\end{aligned}$$

The first term gives

$$\lim_{n\to 1}\frac{\partial}{\partial n}(h_{11}(\theta,n)-h_{11}(-\theta,n)) = -\frac{\pi}{2\cosh^2\frac{\theta}{2}}\left(\frac{f(\theta,1)-f(-\theta,1)}{f(i\pi,1)}\right).$$

The second contribution will come from:

$$\frac{\partial}{\partial n}h_{11}(-\theta+\beta,n) = -\frac{\pi}{2\cosh^2\frac{\theta-\beta}{2}}\left(\frac{f(-\theta+\beta,1)}{f(i\pi,1)}\right).$$

Free Fermion Case
For the free Fermion we know that

$$f(\theta,n) = -i\sinh\frac{\theta}{2n}.$$

Formula (1) can be easily checked numerically against the original sum and there is full agreement for any θ and integer values of n. The derivative simplifies to

$$\lim_{n\to 1}\frac{\partial}{\partial n}\tilde{h}(\theta,n) = \frac{i\pi\tanh\frac{\theta}{2}}{\cosh\frac{\theta}{2}} + \lim_{\epsilon\to 0}\frac{1}{4}\int_{-\infty}^{\infty}\coth\frac{\beta-i\epsilon}{2}\frac{\tanh\frac{\beta-i\epsilon-\theta}{2}}{\cosh\frac{\beta-i\epsilon-\theta}{2}}d\beta. \tag{2}$$

Note that in the integral above, the integration variable β is shifted by a small amount $-i\epsilon$ as in the original contour we wanted to include the value $j=0$ in the sum. If we take this into account it can actually be computed explicitly and the final answer is

$$\lim_{n\to 1}\frac{\partial}{\partial n}\tilde{h}(\theta,n) \;=\; \frac{i\pi\tanh\frac{\theta}{2}}{2\cosh\frac{\theta}{2}} + \frac{\pi}{2}\mathrm{sech}^2\frac{\theta}{2}. \tag{3}$$

Free Boson Case
For the free Boson we know that

$$f(\theta,n) = 1.$$

The S-matrix is 1 so there is no contribution from the integral and no contribution from the term $f(\theta,1)-f(-\theta,1)$ so we get

$$\lim_{n\to 1}\frac{\partial}{\partial n}\tilde{f}(\theta,n) = 0.$$